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Optimal Estimation with Two Process Models and No Measurements

by James M Maley

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1. Introduction

Typically, state estimators are designed to blend the information present in a model of how the states progress through time, called the process model, with measurements of the system outputs. For example, consider the case of a linear discrete-time Kalman filter for estimating a linear time-invariant system with a process model given by:

$$\mathbf{x}_k = \mathbf{F}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_{k-1} + \mathbf{w}_{k-1} . \tag{1}$$

Where the noise input $\mathbf{w}_k$ is a white random sequence and the deterministic input vector $\mathbf{u}_{k-1}$ is known. This model is used to form the state prediction $\hat{\mathbf{x}}_{k+1}^-$ from the last corrected state estimate $\hat{\mathbf{x}}_{k-1}^+$. That is:

$$\hat{\mathbf{x}}_k^- = \mathbf{F}\hat{\mathbf{x}}_{k-1}^+ + \mathbf{B}\mathbf{u}_{k-1} . \tag{2}$$

The Kalman filter seeks to blend this prediction with a measurement of the current state, which is corrupted by the noise term $\boldsymbol{\upsilon}_k$:

$$\mathbf{z}_k = \mathbf{H}\mathbf{x}_k + \boldsymbol{\upsilon}_k . \tag{3}$$

It accomplishes this task by computing a Kalman gain $\mathbf{K}$ so that when the corrected state is calculated via:

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}\left(\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_k^-\right) , \tag{4}$$

the covariance of the new estimate error is minimized.[1] In some applications however, there are no measurements present, but there is more than one way to predict the state based on the previous estimate. Suppose there is a second process model:

$$\mathbf{x}_k = \mathbf{G}\mathbf{x}_{k-1} + \mathbf{E}\mathbf{n}_{k-1} + \mathbf{v}_{k-1} , \tag{5}$$

with noisy inputs $\mathbf{v}_k$ and known deterministic inputs $\mathbf{n}_{k-1}$. The problem is to combine the 2 predictions $\hat{\mathbf{x}}_{k,f}^- = \mathbf{F}\hat{\mathbf{x}}_{k-1}^+ + \mathbf{B}\mathbf{u}_{k-1}$ and $\hat{\mathbf{x}}_{k,g}^- = \mathbf{G}\hat{\mathbf{x}}_{k-1}^+ + \mathbf{E}\mathbf{n}_{k-1}$ so that the combined state estimate has the smallest variance. This situation does not fit the format for which observability is defined. At no point is the true state $\mathbf{x}_k$ mapped to a measurement. This is different from the multiple model estimation problem, which uses measurement residuals to make an optimal guess at which process model is correct. The problem discussed here is most similar to decentralized Kalman filtering, which combines several independent Kalman filter estimates, but all of the Kalman filters use the same propagation equations. The main issue tackled in decentralized filtering is dealing with networks of sensors that may or may not have access to the

“central” state estimate and covariance matrix and the various timing and data transfer issues associated with such systems.

This problem could present itself in navigation applications. In the example used here, one process model comes from projectile flight dynamics, while another process model comes from inertial measurement unit (IMU) integration. The primary motivation for this work is to improve on the methodologies used in Fairfax and Fresconi,[2] which addresses this application in more detail. Another application would be the blending of 2 uncorrected inertial measurement units, which is becoming more prevalent as the size and cost of microelectromechanical system (MEMS) inertial devices continues to decrease.

In the next section, an estimator is derived for the simple linear 2-process-model system described in the introduction. In Section 3 a simulation example is given that blends a dynamic model of a ballistic projectile with an IMU output. Section 4 discusses the conclusions and future work.

2. Estimator Derivation

The simple scenario involves process model 1, which is given above in Eq. 1, and has an initial state error covariance of:

$$\mathbf{P}_{k-1,f}^{+} = \mathbf{E}\left[\mathbf{e}_{k-1,f}\mathbf{e}_{k-1,f}^{T}\right], \tag{6}$$

and process noise that is normally distributed as:

$$\mathbf{w}_{k-1} \sim N\left(\mathbf{0},\mathbf{Q}_{w}\right). \tag{7}$$

The error vector $\mathbf{e}_k$ is defined as the true state minus the estimated state:

$$\mathbf{e}_{k} = \mathbf{x}_{k} - \hat{\mathbf{x}}_{k}. \tag{8}$$

This must be combined with process model 2, which is given above in Eq. 5 and has an initial state error covariance of:

$$\mathbf{P}_{k-1,g}^{+} = \mathbf{E}\left[\mathbf{e}_{k-1,g}\mathbf{e}_{k-1,g}^{T}\right], \tag{9}$$

and process noise that is normally distributed as:

$$\mathbf{v}_{k-1} \sim N\left(\mathbf{0},\mathbf{Q}_{v}\right). \tag{10}$$

An estimator that blends these 2 process models is hypothesized to have the form:

$$\hat{\mathbf{x}}_{k}^{+} = \mathbf{F}\hat{\mathbf{x}}_{k-1}^{+} + \mathbf{B}\mathbf{u}_{k-1} + \mathbf{K}\left(\mathbf{G}\hat{\mathbf{x}}_{k-1}^{+} + \mathbf{E}\mathbf{n}_{k-1} - \mathbf{F}\hat{\mathbf{x}}_{k-1}^{+} - \mathbf{B}\mathbf{u}_{k-1}\right). \tag{11}$$

The goal is then to find what $\mathbf{K}$ must be to minimize $E\left[\mathbf{e}_k^{+}\mathbf{e}_k^{+T}\right]$. The expression for the estimator error can be determined by subtracting the estimated state from the true state:

$$\begin{aligned}\mathbf{e}_k^{+} &= \mathbf{x}_k - \hat{\mathbf{x}}_k^{+} \\ &= \mathbf{F}\mathbf{e}_{k-1,f}^{+} + \mathbf{w}_k - \mathbf{K}\left(\mathbf{G}\hat{\mathbf{x}}_{k-1}^{+} + \mathbf{E}\mathbf{n}_{k-1} - \mathbf{F}\hat{\mathbf{x}}_{k-1}^{+} - \mathbf{B}\mathbf{u}_{k-1}\right)\end{aligned}. \tag{12}$$

The term inside the parenthesis can be written in terms of the errors by noting that:

$$-\mathbf{x}_k + \mathbf{x}_k = -\mathbf{G}\mathbf{x}_{k-1} - \mathbf{E}\mathbf{n}_{k-1} - \mathbf{v}_{k-1} + \mathbf{F}\mathbf{x}_{k-1} + \mathbf{B}\mathbf{u}_{k-1} + \mathbf{w}_{k-1} = \mathbf{0}\,. \tag{13}$$

The new expression for the estimator error is:

$$\mathbf{e}_k^{+} = \mathbf{F}\mathbf{e}_{k-1,f}^{+} + \mathbf{w}_{k-1} + \mathbf{K}\left(\mathbf{G}\mathbf{e}_{k-1}^{+} + \mathbf{v}_{k-1} - \mathbf{F}\mathbf{e}_{k-1}^{+} - \mathbf{w}_{k-1}\right). \tag{14}$$

Using the assumption that the process noise from the 2 systems is uncorrelated leads to the simplification:

$$E\left[\mathbf{w}\mathbf{v}^T\right] = E\left[\mathbf{v}\mathbf{w}^T\right] = \mathbf{0}\,. \tag{15}$$

A second assumption is that noise terms are uncorrelated with the current state errors:

$$E\left[\mathbf{e}_k\mathbf{v}_k^T\right] = E\left[\mathbf{e}_k\mathbf{w}_k^T\right] = \mathbf{0}\,. \tag{16}$$

There are 2 situations under which the estimator is being used. The first situation is that the estimator has been continually blending the 2 process models. In this case the errors at the previous step are identical, and have the same covariance, that is:

$$E\left[\mathbf{e}_{k-1}^{+}\mathbf{e}_{k-1}^{+T}\right] = \mathbf{P}_{k-1}^{+}\,. \tag{17}$$

The estimator covariance is then:

$$\begin{aligned}&E\left[\mathbf{e}_k^{+},\mathbf{e}_k^{+T}\right] = \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^T + \mathbf{Q}_w \\ &\ldots + \left(-\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^T - \mathbf{Q}_w + \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{G}^T\right)\mathbf{K}^T + \mathbf{K}\left(-\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^T - \mathbf{Q}_w + \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{F}^T\right). \\ &\ldots + \mathbf{K}\left(\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^T + \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{G}^T - \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{F}^T - \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{G}^T + \mathbf{Q}_w + \mathbf{Q}_v\right)\mathbf{K}^T\end{aligned} \tag{18}$$

The value of $\mathbf{K}$ is determined by taking the derivative of the trace of the estimator covariance and solving for the zero slope condition, that is:

$$\begin{aligned}&\frac{d\left[tr\left(E\left[\mathbf{e}_k^{+}\mathbf{e}_k^{+T}\right]\right)\right]}{d\mathbf{N}} = 2\mathbf{K}\left(\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^T + \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{G}^T - \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{F}^T - \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{G}^T + \mathbf{Q}_w + \mathbf{Q}_v\right). \\ &\ldots - 2\left(\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^T + \mathbf{Q}_w - \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{G}^T\right)\end{aligned} \tag{19}$$

The estimator gain is therefore:

$$\mathbf{K} = \left(\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^{T} + \mathbf{Q}_{w} - \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{G}^{T}\right)\left(\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^{T} + \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{G}^{T} - \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{F}^{T} - \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{G}^{T} + \mathbf{Q}_{w} + \mathbf{Q}_{v}\right)^{-1}. \quad (20)$$

Substituting this into parts of Eq. 12 simplifies the estimator covariance expression to:

$$\mathbf{P}_{k}^{+} = \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^{T} + \mathbf{Q}_{w} - \mathbf{K}\left(\mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^{T} + \mathbf{Q}_{w} - \mathbf{G}\mathbf{P}_{k-1}^{+}\mathbf{F}^{T}\right). \quad (21)$$

In some cases the 2 process models are identical, such as when blending 2 IMUs. This simplification reduces the gain to a simple function of the process noise covariance matrices:

$$\mathbf{K} = \mathbf{Q}_{w}\left(\mathbf{Q}_{w} + \mathbf{Q}_{v}\right)^{-1}. \quad (22)$$

The covariance matrix update is then:

$$\mathbf{P}_{k}^{+} = \mathbf{F}\mathbf{P}_{k-1}^{+}\mathbf{F}^{T} + \mathbf{Q}_{w} - \mathbf{Q}_{w}\left(\mathbf{Q}_{w} + \mathbf{Q}_{v}\right)^{-1}\mathbf{Q}_{w}. \quad (23)$$

The second situation is that the process models have been running independently from each other up until the current time. In this case, their errors are assumed to be uncorrelated:[3]

$$E\left[\mathbf{e}_{k-1,f}^{-}\mathbf{e}_{k-1,g}^{-T}\right] = E\left[\mathbf{e}_{k-1,g}^{-}\mathbf{e}_{k-1,f}^{-T}\right] = \mathbf{0}. \quad (24)$$

The estimator gain is:

$$\begin{aligned}\mathbf{K} &= \left(\mathbf{F}\mathbf{P}_{k-1,f}^{-}\mathbf{F}^{T} + \mathbf{Q}_{w}\right)\left(\mathbf{F}\mathbf{P}_{k-1,f}^{-}\mathbf{F}^{T} + \mathbf{G}\mathbf{P}_{k-1,g}^{-}\mathbf{G}^{T} + \mathbf{Q}_{w} + \mathbf{Q}_{v}\right)^{-1} \\ &= \mathbf{P}_{k,f}^{-}\left(\mathbf{P}_{k,f}^{-} + \mathbf{P}_{k,g}^{-}\right)^{-1}\end{aligned}. \quad (25)$$

The covariance update is:

$$\mathbf{P}_{k}^{+} = \mathbf{P}_{k,f}^{-} - \mathbf{K}\mathbf{P}_{k,f}^{-}. \quad (26)$$

3. Simulation Example

To verify the results in the above section, a simulation experiment was performed on 2 simple process models with known statistics. The first is a point mass flight dynamic model of a ballistic projectile that is driven by a white noise forcing function. The next model is a simplified inertial measurement model, in which the accelerations of the projectile are numerically integrated after adding white noise to them.

3.1 Flight Dynamic Model

The dynamic model for the ballistic projectile has the form:

$$\begin{Bmatrix} \ddot{x} \\ \ddot{z} \end{Bmatrix} = -\frac{\rho\sqrt{\dot{x}^2+\dot{z}^2}\,AC_{x0}}{2m}\begin{Bmatrix} \dot{x} \\ \dot{z} \end{Bmatrix} + \begin{Bmatrix} 0 \\ g \end{Bmatrix} + \mathbf{w}\,. \tag{27}$$

Where the following constants are defined below in the table:

Table Nomenclature and values used for simple dynamic model

Name	Symbol	Value Used	Units
Air Density	ρ	1.2	kg/m^3
Cross-Sectional Area	A	$\pi 0.155^2/4$	m^2
Axial Force Coefficient	C_{x0}	0.2	nondimensional
Mass	m	40	kg
Gravitational Acceleration	g	9.81	m/s^2

The initial conditions for the projectile state vector are given by:

$$\mathbf{x}_0 = \begin{Bmatrix} x \\ z \\ \dot{x} \\ \dot{z} \end{Bmatrix} = \begin{Bmatrix} 0\ m \\ 0\ m \\ 500\,m/s \\ -500\,m/s \end{Bmatrix}. \tag{28}$$

The dynamic model is nonlinear. Normally the equations of motion from Eq. 27 would be integrated numerically with a Runge-Kutte integration algorithm. However, to keep the statistics more “known” in both models, a simple Euler integration algorithm is used to form the discrete time model:

$$\begin{Bmatrix} x_k \\ z_k \\ \dot{x}_k \\ \dot{z}_k \end{Bmatrix} = \begin{Bmatrix} x_{k-1} \\ z_{k-1} \\ \dot{x}_{k-1} \\ \dot{z}_{k-1} \end{Bmatrix} + \left\{ \begin{Bmatrix} \dot{x}_{k-1} \\ \dot{z}_{k-1} \\ \frac{\rho\sqrt{\dot{x}_{k-1}^2+\dot{z}_{k-1}^2}\,AC_{x0}}{2m}\dot{x}_{k-1} \\ \frac{\rho\sqrt{\dot{x}_{k-1}^2+\dot{z}_{k-1}^2}\,AC_{x0}}{2m}\dot{z}_{k-1}+g \end{Bmatrix} + \mathbf{w}_{k-1} \right\}\Delta t \quad . \tag{29}$$

In order to propagate the covariance of the model, the state transition matrix is formed by taking the jacobian of the state-propagation equations with respect to the states:

$$\mathbf{F}(\Delta t)_{k-1} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1-\frac{2AC_{x0}\Delta t\rho\dot{x}_{k-1}^2 + AC_{x0}\Delta t\rho\dot{z}_{k-1}^2}{2m\sqrt{\dot{x}_{k-1}^2+\dot{z}_{k-1}^2}} & -\frac{AC_{x0}\Delta t\rho\dot{x}_{k-1}\dot{z}_{k-1}}{2m\sqrt{\dot{x}_{k-1}^2+\dot{z}_{k-1}^2}} \\ 0 & 0 & -\frac{AC_{x0}\Delta t\rho\dot{x}_{k-1}\dot{z}_{k-1}}{2m\sqrt{\dot{x}_{k-1}^2+\dot{z}_{k-1}^2}} & 1-\frac{AC_{x0}\Delta t\rho\dot{x}_{k-1}^2 + 2AC_{x0}\Delta t\rho\dot{z}_{k-1}^2}{2m\sqrt{\dot{x}_{k-1}^2+\dot{z}_{k-1}^2}} \end{bmatrix}. \tag{30}$$

The process noise covariance matrix was determined by using the following equation, although the resulting expression is too lengthy to display:

$$\mathbf{Q}_w = \int_0^{\Delta t} \mathbf{F}(\Delta t - \tau)\mathbf{Q}_c \mathbf{F}^T(\Delta t - \tau)\, d\tau \,. \tag{31}$$

The $\mathbf{Q}_c$ matrix is given by:[4]

$$\mathbf{Q}_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_w^2 & 0 \\ 0 & 0 & 0 & \sigma_w^2 \end{bmatrix} \Delta t \,. \tag{32}$$

An example 10-second trajectory is plotted in Fig. 1.

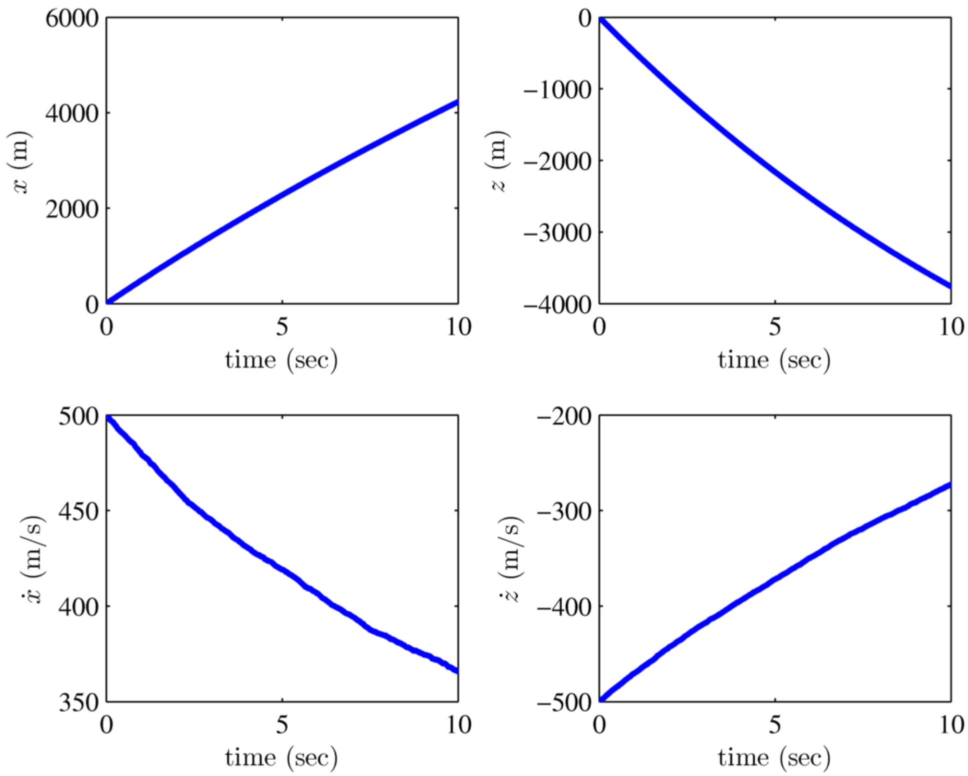

Fig. 1 Example dynamic model ballistic trajectory

An ensemble of 100 "true" trajectories was run with $\sigma_w = 10\,\mathrm{m/s}^2$ by integrating Eq. 29. An estimated trajectory was run by integrating:

$$\hat{\mathbf{x}}_{k,f} = \hat{\mathbf{x}}_{k-1,f} + \left\{ \begin{matrix} \hat{\dot{x}}_{k-1} \\ \hat{\dot{z}}_{k-1} \\ \dfrac{\rho\sqrt{\hat{\dot{x}}_{k-1}^2 + \hat{\dot{z}}_{k-1}^2}\, A C_{x0}}{2m}\, \hat{\dot{x}}_{k-1} \\ \dfrac{\rho\sqrt{\hat{\dot{x}}_{k-1}^2 + \hat{\dot{z}}_{k-1}^2}\, A C_{x0}}{2m}\, \hat{\dot{z}}_{k-1} + g \end{matrix} \right\} \Delta t \,. \tag{33}$$

The resulting root sum squared (rss) deviation from the nominal trajectory run with zero noise was compared to the 1σ values taken from the diagonals of the propagated error covariance matrix. The 2 results are shown in Fig. 2. The figure shows that the covariance propagation is modeled consistently with the random process being simulated.

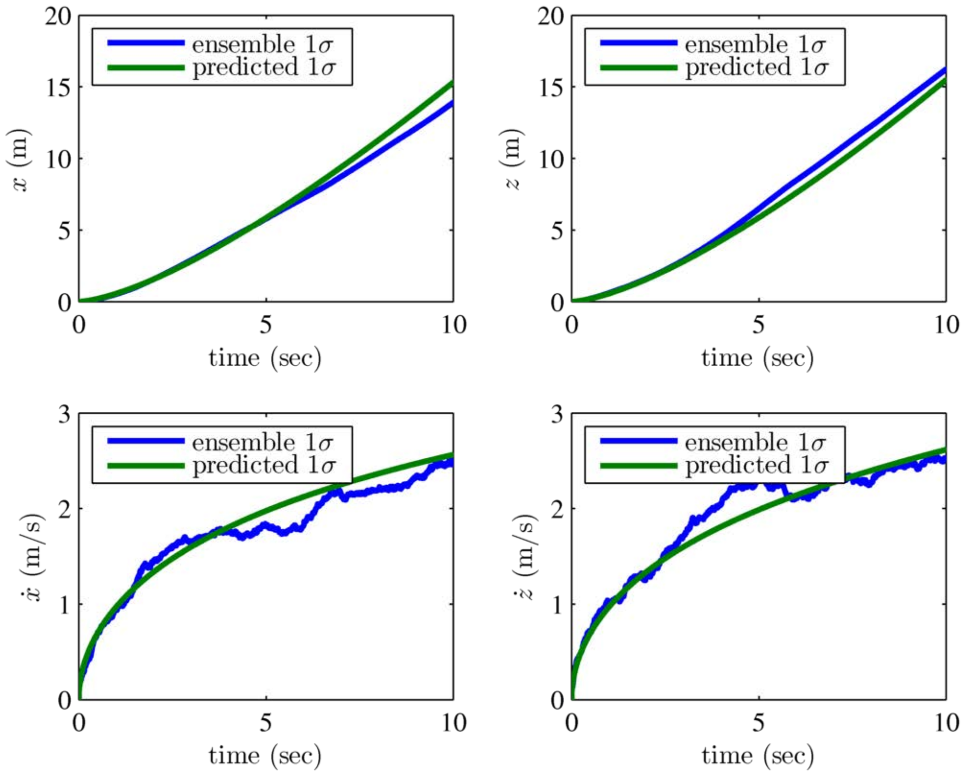

Fig. 2 Predicted vs. measured error standard deviation for the dynamic model

3.2 Inertial Measurement Model

The inertial measurement model consists of integrating accelerometer signals $\hat{a}_x$ and $\hat{a}_z$:

$$\begin{Bmatrix} \ddot{x} \\ \ddot{z} \end{Bmatrix} = \begin{Bmatrix} \hat{a}_x \\ \hat{a}_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ g \end{Bmatrix} + \mathbf{v} \,. \tag{34}$$

The true accelerometer values a_x and a_z are generated from the dynamic model equations:

$$\begin{Bmatrix} a_x \\ a_z \end{Bmatrix} = -\frac{\rho\sqrt{\dot{x}_k^2 + \dot{z}_k^2}\,AC_{x0}}{2m} \begin{Bmatrix} \dot{x}_k \\ \dot{z}_k \end{Bmatrix} + \mathbf{w}_k \,. \tag{35}$$

Each simulation of the dynamic model therefore also produces a noiseless accelerometer trajectory, which is then corrupted with noise:

$$\begin{Bmatrix} \hat{a}_{x,k} \\ \hat{a}_{z,k} \end{Bmatrix} = \begin{Bmatrix} a_{x,k} \\ a_{z,k} \end{Bmatrix} + \begin{Bmatrix} v_{x,k} \\ v_{z,k} \end{Bmatrix}, \tag{36}$$

where each noise term $v_{x,k}$ and $v_{z,k}$ is normally distributed with zero mean and standard deviation σ_v.

Although the accelerometers are generated using the dynamic model equations, in practice the dynamic model is usually not known or utilized when integrating an IMU. Using this modeling approach, the accelerometers act as deterministic inputs:

$$\mathbf{x}_{k,g} = \mathbf{G}(\Delta t)\mathbf{x}_{k-1,g} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{Bmatrix} \hat{a}_x \\ \hat{a}_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \Delta t g \end{Bmatrix} + \mathbf{v}_k, \tag{37}$$

$$\mathbf{G}(\Delta t) = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

Similar to the dynamic model case, the discrete time process noise covariance is calculated from:

$$\mathbf{Q}_v = \int_0^{\Delta t} \mathbf{G}(\Delta t - \tau)\mathbf{Q}_c \mathbf{G}^T(\Delta t - \tau)\, d\tau. \tag{39}$$

The $\mathbf{Q}_c$ matrix is given by:

$$\mathbf{Q}_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_v^2 & 0 \\ 0 & 0 & 0 & \sigma_v^2 \end{bmatrix} \Delta t. \tag{40}$$

Each estimated trajectory from the IMU model is calculated by integrating:

$$\hat{\mathbf{x}}_{k,g} = \mathbf{G}(\Delta t)\hat{\mathbf{x}}_{k-1,g} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{Bmatrix} \hat{a}_x \\ \hat{a}_z \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \Delta t g \end{Bmatrix}. \tag{41}$$

The ensemble standard deviation is compared to the standard deviation from the propagated covariance when $\sigma_v = 10$ m/s^2. The results are shown in Fig. 3.

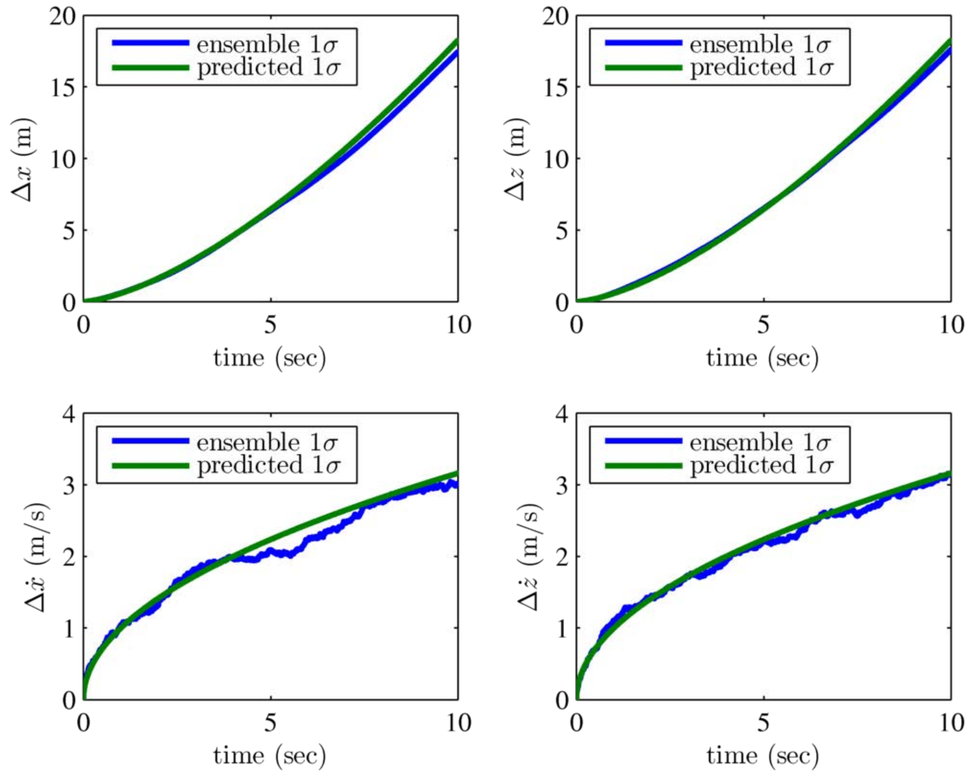

Fig. 3 Predicted vs. measured error standard deviation for the IMU model

3.3 Estimator Performance

The multiple process model estimator (MPME) from Section 2 was used to combine the predicted dynamic model trajectory from Eq. 33 with the predicted IMU trajectory from Eq. 41. Figure 4 verifies that the covariance update given in Eq. 15 is consistent with the actual measured error covariance.

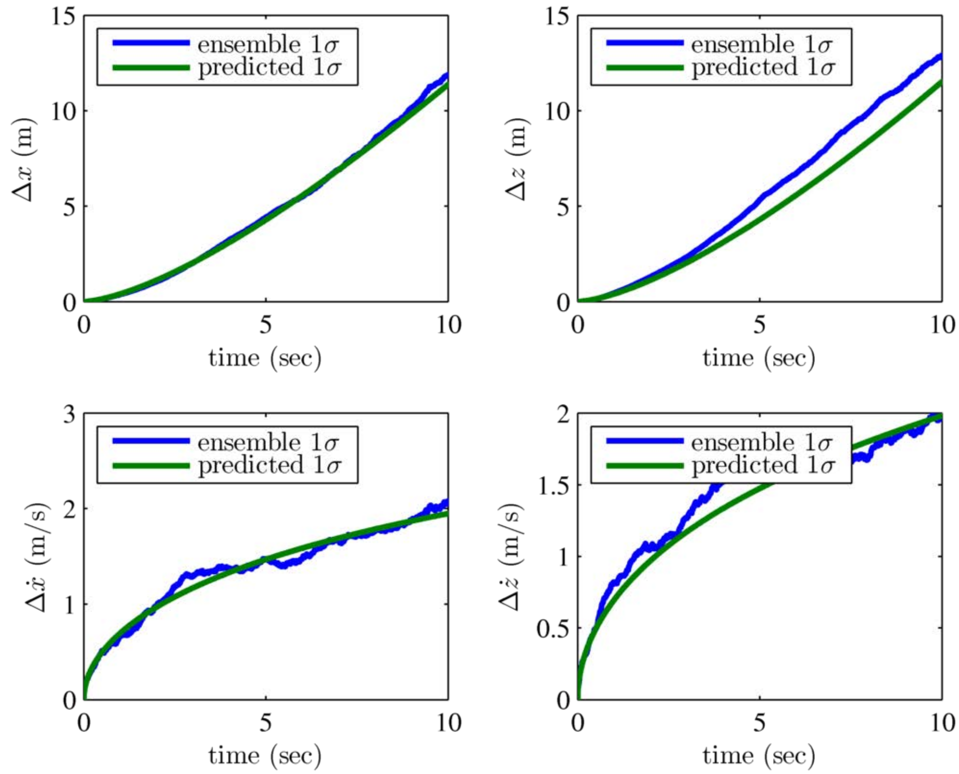

Fig. 4 Predicted vs. measured error standard deviation for the state estimator

Fig. 5 shows that for this example, the MPME produces a lower ensemble variance than either independent process model. The simulation example presented in this section was also run with different values of σ_w and σ_v . In all cases the estimator has less error than either independent process model on average, but the improvement is more noticeable the closer the 2 noise values are to each other. The MPME was also run with the process models flipped—that is, $\hat{\mathbf{x}}_k^+ = \mathbf{G}\hat{\mathbf{x}}_{k-1}^+ + \mathbf{E}\mathbf{n}_{k-1} + \mathbf{K}\left(\mathbf{F}\hat{\mathbf{x}}_{k-1}^+ + \mathbf{B}\mathbf{u}_{k-1} - \mathbf{G}\hat{\mathbf{x}}_{k-1}^+ - \mathbf{E}\mathbf{n}_{k-1}\right)$—with identical results.[5]

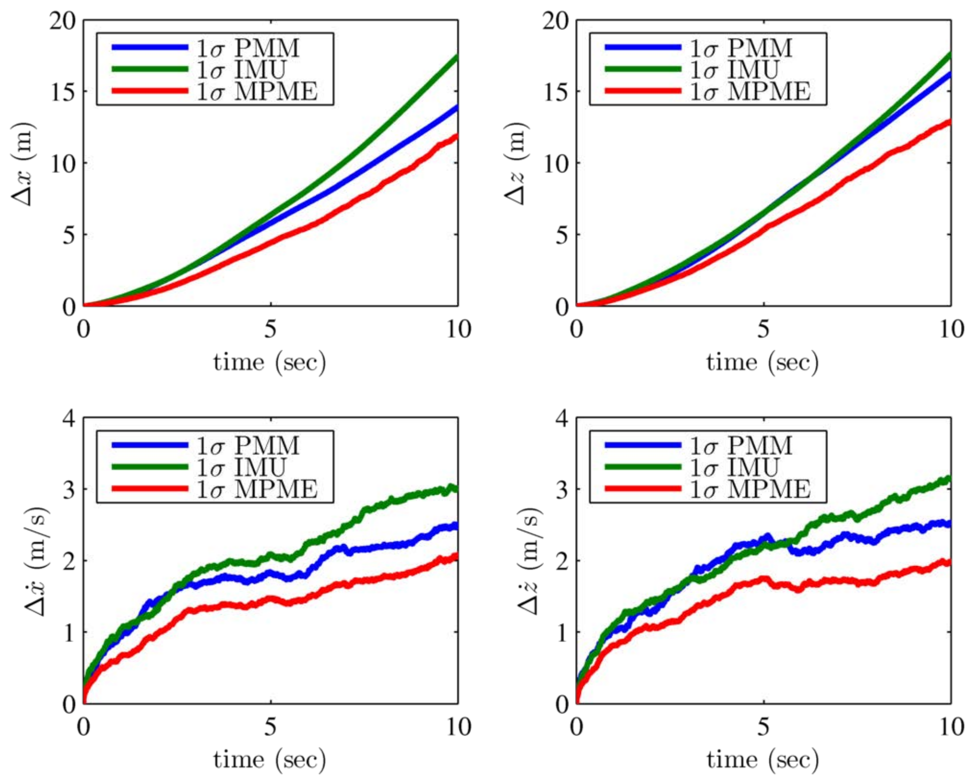

Fig. 5 Comparison of the ensemble standard deviation values from the 2 independent process models and the combined state estimate

4. Conclusions

A multiple process model estimator was designed to blend the predictions of 2 separate discrete-time process models when no measurements are available to obtain an optimal state estimate. The estimator was validated with a simple example with known statistics in which one process model based on projectile flight dynamics was combined with a separate process model based on inertial measurement integration. Although the systems were greatly simplified, they were useful for verifying the estimator. It was found that on average, the estimator produces state estimates with lower variance than either independent process model. These results should prove useful in situations where the process is truly random and the statistical distributions of the forcing functions are known.

The case where the dynamic model is not actually driven by white noise was not addressed here. It is expected that the marginal improvements gained by blending 2 noisy process models will be lost if either of the models includes deterministic modeling errors.

5. References and Notes

1. Brown RG, Hwang PYC. Introduction to random signals and applied Kalman filtering, 3rd Ed., John Wiley and Sons Inc., Hoboken, NJ, 1997.

2. Fairfax L, Fresconi F. Position estimation for projectiles using low-cost sensors and flight dynamics. Aberdeen Proving Ground (MD): Army Research Laboratory (US); 2012. Report No.: ARL-TR-5994. Also available at http://www.arl.army.mil/www/default.cfm?technical_report=5994.

3. The - superscript denotes that the errors have not been through any kind of "corrector" step.

4. The Δt term comes from the fact that Euler integration is being used to propagate the model.

5. The entire systems where switched including process noise covariance matrices. The point of the exercise was to show that it doesn't matter which process model is considered model 1 and which is model 2.

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